* Matrix Balancing for Input-Output Tables: Application of the Method of Lagrange Multipliers

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* What is matrix balancing?
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Matrix Balancing

$$\sum_{j=1}^{m} x_{ij} = X_{R_i} \quad (i = 1 \cdots n)$$

$$\sum_{i=1}^{n} x_{ij} = X_{C_j} \quad (j = 1 \cdots m)$$

	1	…	m	
1	x_{11}	…	x_{1m}	X_{R_1}
⋮	⋮	⋱	⋮	⋮
n	x_{n1}	…	x_{nm}	X_{R_n}
	X_{C_1}	…	X_{C_m}	

X_{R_i} and X_{C_j} : Constraint Conditions

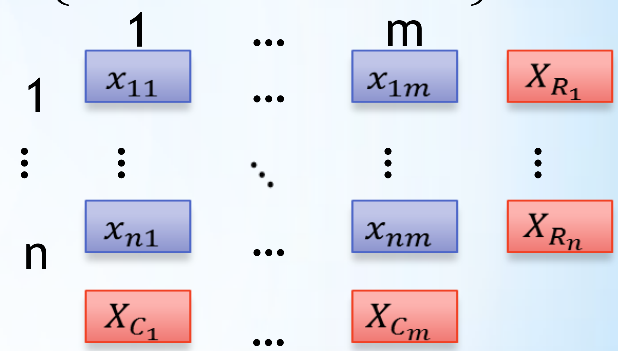
The objective of matrix balancing is to determine x_{ij} consistent with the constraint conditions X_{R_i} and X_{C_j}.

* Method of Lagrange Multipliers

min : $$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left\{ w_{R_{ij}}\left(\frac{x_{ij}}{X_{R_i}}-a_{R_{ij}}\right)^2\right\}+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left\{ w_{C_{ij}}\left(\frac{x_{ij}}{X_{C_j}}-a_{C_{ij}}\right)^2\right\}$$

s.t. : $$\sum_{j=1}^{m} x_{ij} = X_{R_i} \quad (i=1\cdots n)$$

$$\sum_{i=1}^{n} x_{ij} = X_{C_j} \quad (j=1\cdots m)$$



$a_{R_{ij}}$ and $a_{C_{ij}}$: Benchmarks $w_{R_{ij}}$ and $w_{C_{ij}}$: Weights

If $w_{R_{ij}} = 1/a^2{}_{R_{ij}}$ and $w_{C_{ij}} = 1/a^2{}_{C_{ij}}$, the method is called KEO-RAS method. (KEO: Keio Economic Observatory)

Kuroda, M (1988), A Method of estimation for updating transaction matrix in the input-output relationships, Uno, K. and Shishido, S. eds., *Statistical Data Bank Systems, Socio-Economic Database and Model Building in Japan*, Amsterdam, pp.128-148.

* KEO-RAS Method

min :

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{x_{ij}/X_{R_i}}{a_{R_{ij}}}-1\right)^2+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{x_{ij}/X_{C_j}}{a_{C_{ij}}}-1\right)^2$$

s.t. :

$$\sum_{j=1}^{m}x_{ij}=X_{R_i}\quad(i=1\cdots n)$$

$$\sum_{i=1}^{n}x_{ij}=X_{C_j}\quad(j=1\cdots m)$$

The Lagrangian function is;

$$L=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{x_{ij}/X_{R_i}}{a_{R_{ij}}}-1\right)^2+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{x_{ij}/X_{C_j}}{a_{C_{ij}}}-1\right)^2$$

$$+\sum_{i=1}^{n}\lambda_{R_i}\left(X_{R_i}-\sum_{j=1}^{m}x_{ij}\right)+\sum_{j=1}^{m}\lambda_{C_j}\left(X_{C_j}-\sum_{i=1}^{n}x_{ij}\right)$$

(λ_{R_i} and λ_{C_i} : Lagrange multipliers)

* KEO-RAS Method

Necessary Conditions for Minimization

$$\frac{\partial L}{\partial x_{ij}} = \frac{1}{a_{R_{ij}} X_{R_i}} \left(\frac{x_{ij}}{a_{R_{ij}} X_{R_i}} - 1 \right) + \frac{1}{a_{C_{ij}} X_{C_j}} \left(\frac{x_{ij}}{a_{C_{ij}} X_{C_j}} - 1 \right) - \lambda_{R_i} - \lambda_{C_j} = 0$$

$$\frac{\partial L}{\partial \lambda_{R_i}} = \left(X_{R_i} - \sum_{j=1}^{m} x_{ij} \right) = 0$$

$$\frac{\partial L}{\partial \lambda_{C_j}} = \left(X_{C_j} - \sum_{i=1}^{n} x_{ij} \right) = 0$$

$$(i = 1 \cdots n,\ j = 1 \cdots m)$$

$$x_{ij} = \frac{a_{R_{ij}}^2 X_{R_i}^2 a_{C_{ij}}^2 X_{C_j}^2}{a_{R_{ij}}^2 X_{R_i}^2 + a_{C_{ij}}^2 X_{C_j}^2} \left(\frac{1}{a_{R_{ij}} X_{R_i}} + \frac{1}{a_{C_{ij}} X_{C_j}} + \lambda_{R_i} + \lambda_{C_j} \right)$$

* KEO-RAS Method

$$\lambda_{R_i}\sum_{j=1}^{m}s_{ij}+\sum_{j=1}^{m}\lambda_{C_j}s_{ij}=X_{R_i}-\sum_{j=1}^{m}s_{ij}t_{ij}\qquad (i=1\cdots n)$$

$$\sum_{i=1}^{n}\lambda_{R_i}s_{ij}+\lambda_{C_j}\sum_{i=1}^{n}s_{ij}=X_{C_j}-\sum_{i=1}^{n}s_{ij}t_{ij}\qquad (j=1\cdots m)$$

Here,

$$s_{ij}=\frac{a_{R_{ij}}^{2}X_{R_i}^{2}a_{C_{ij}}^{2}X_{C_j}^{2}}{a_{R_{ij}}^{2}X_{R_i}^{2}+a_{C_{ij}}^{2}X_{C_j}^{2}}\qquad t_{ij}=\frac{1}{a_{R_{ij}}X_{R_i}}+\frac{1}{a_{C_{ij}}X_{C_j}}$$

$$\begin{bmatrix}
\sum_{j=1}^{m}s_{1j} & & 0 & s_{11} & \cdots & s_{1m}\\
 & \ddots & & \vdots & \ddots & \vdots\\
0 & & \sum_{j=1}^{m}s_{nj} & s_{n1} & \cdots & s_{nm}\\
s_{11} & \cdots & s_{n1} & \sum_{i=1}^{n}s_{i1} & & 0\\
\vdots & \ddots & \vdots & & \ddots & \\
s_{1m} & \cdots & s_{nm} & 0 & & \sum_{i=1}^{n}s_{im}
\end{bmatrix}
\begin{bmatrix}\lambda_{R_1}\\ \vdots\\ \lambda_{R_n}\\ \lambda_{C_1}\\ \vdots\\ \lambda_{C_m}\end{bmatrix}
=
\begin{bmatrix}
X_{R_1}-\sum_{j=1}^{m}s_{1j}t_{1j}\\
\vdots\\
X_{R_n}-\sum_{j=1}^{m}s_{nj}t_{nj}\\
X_{C_1}-\sum_{i=1}^{n}s_{i1}t_{i1}\\
\vdots\\
X_{C_m}-\sum_{i=1}^{n}s_{im}t_{im}
\end{bmatrix}$$

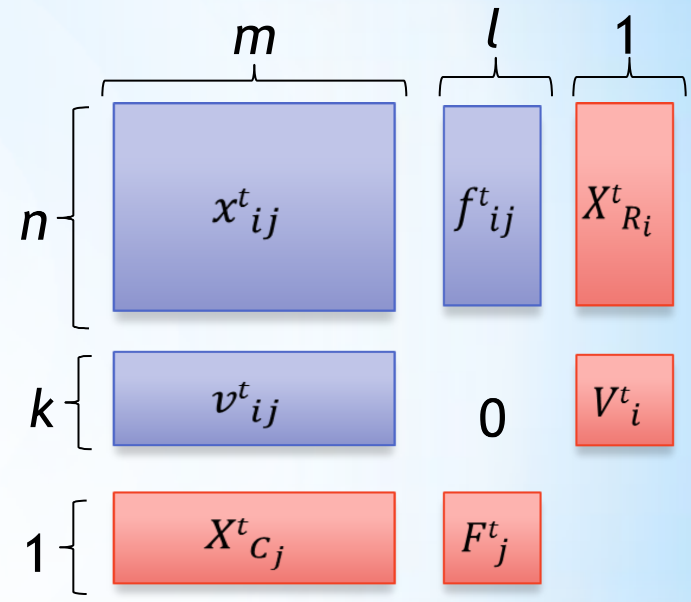
Matrix Balancing for Input-Output Tables

min :

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{x_{ij}^{t}/X_{C_j}^{t}}{x_{ij}^{0}/X_{C_j}^{0}}-1\right)^2+\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{m}\left(\frac{v_{ij}^{t}/X_{C_j}^{t}}{v_{ij}^{0}/X_{C_j}^{0}}-1\right)^2$$

$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{l}\left(\frac{f_{ij}^{t}/F_{j}^{t}}{f_{ij}^{0}/F_{j}^{0}}-1\right)^2+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{x_{ij}^{t}/X_{R_i}^{t}}{x_{ij}^{0}/X_{R_i}^{0}}-1\right)^2$$

$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{l}\left(\frac{f_{ij}^{t}/X_{R_i}^{t}}{f_{ij}^{0}/X_{R_i}^{0}}-1\right)^2+\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{m}\left(\frac{v_{ij}^{t}/V_{i}^{t}}{v_{ij}^{0}/V_{i}^{0}}-1\right)^2$$



s.t. :

$$\sum_{i=1}^{n}x_{ij}^{t}+\sum_{i=1}^{k}v_{ij}^{t}=X_{C_j}^{t}\quad(j=1\cdots m)$$

$$\sum_{i=1}^{n}f_{ij}^{t}=F_{j}^{t}\quad(j=1\cdots l)$$

$$\sum_{j=1}^{m}x_{ij}^{t}+\sum_{j=1}^{l}f_{ij}^{t}=X_{R_i}^{t}\quad(i=1\cdots n)$$

$$\sum_{j=1}^{m}v_{ij}^{t}=V_{i}^{t}\quad(i=1\cdots k)$$

* Problems

* Fluctuation in Prices

* Nominal input coefficients are changed by fluctuation in prices.

Input coefficients should be defined based on the real value (not nominal value).

* Instability of Output Coefficients

* Export changes regardless of the domestic economic situation.

Matrix balancing should be conducted except for export.

* Derivation of Real Input Coefficients

$$x_{ij}^{t} = x_{ij}^{t(D)} + x_{ij}^{t(M)} \qquad (i = 1 \cdots n,\ j = 1 \cdots m)$$

$x_{ij}^{t(D)}$: Domestic Products

$x_{ij}^{t(M)}$: Imported Products

$$P_{ij}^{t(C)} = \frac{x_{ij}^{0(D)}}{x_{ij}^{0(D)} + x_{ij}^{0(M)}} P_{i}^{t(D)} + \frac{x_{ij}^{0(M)}}{x_{ij}^{0(D)} + x_{ij}^{0(M)}} P_{i}^{t(M)}$$

$P_{ij}^{t(C)}$: Composite Price Index

$P_{i}^{t(M)}$: Import Price Index

$P_{i}^{t(D)}$: Domestic Price Index

$$P_{i}^{0(C)} = P_{i}^{0(D)} = P_{i}^{0(M)} = 1$$

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{x_{ij}^{t} / X_{C_j}^{t}}{x_{ij}^{0} / X_{C_j}^{0}} - 1\right)^2 \Rightarrow \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{\dfrac{x_{ij}^{t} / P_{ij}^{(C)}}{X_{C_j}^{t} / P_{j}^{(D)}}}{x_{ij}^{0} / X_{C_j}^{0}} - 1\right)^2$$

Effects of Export on Output Coefficients

$$\frac{x_{ij}^{t}}{X_{R_i}^{t}} = \frac{x_{ij}^{t}}{\sum_{j=1}^{m} x_{ij}^{t} + \sum_{j=1}^{l-2} f_{ij}^{t} + e_i^{t} - m_i^{t}}$$

- If e_i changes significantly, x_{ij}/X_{R_i} will also change.
- e_i and m_i can be observed easily from the trade statistics.

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{x_{ij}^{t} / X_{R_i}^{t}}{x_{ij}^{0} / X_{R_i}^{0}} - 1\right)^2 \Rightarrow \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{x_{ij}^{t} /(X_{R_i}^{t} - e_i^{t} + m_i^{t})}{x_{ij}^{0} /(X_{R_i}^{0} - e_i^{0} + m_i^{0})} - 1\right)^2$$

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{l}\left(\frac{f_{ij}^{t} / X_{R_i}^{t}}{f_{ij}^{0} / X_{R_i}^{0}} - 1\right)^2 \Rightarrow \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{l-2}\left(\frac{f_{ij}^{t} /(X_{R_i}^{t} - e_i^{t} + m_i^{t})}{f_{ij}^{0} /(X_{R_i}^{0} - e_i^{0} + m_i^{0})} - 1\right)^2$$

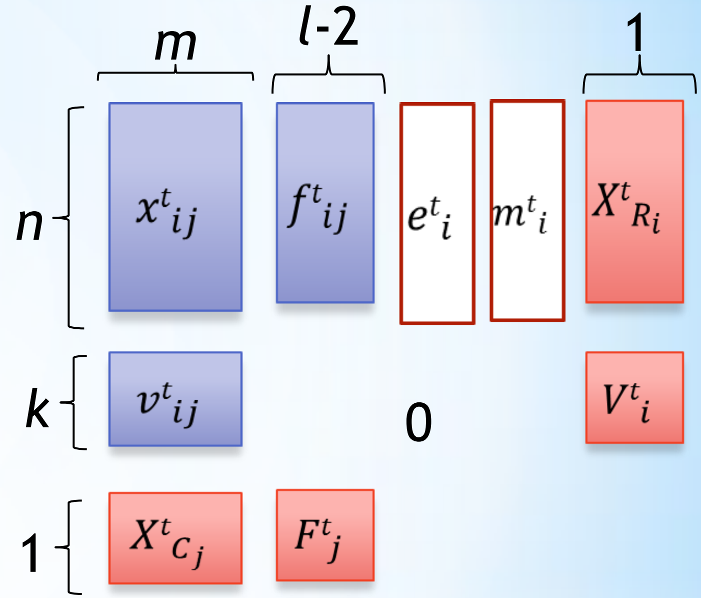
* Matrix Balancing for Input-Output Tables

min :

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{\dfrac{x_{ij}^{t}/P_{ij}^{(C)}}{X_{C_j}^{t}/P_{j}^{(J)}}}{x_{C_{ij}}^{0}/X_{C_j}^{0}}-1\right)^2+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\frac{x_{ij}^{t}/(X_{R_i}^{t}-e_i^t+m_i^t)}{x_{ij}^{0}/(X_{R_i}^{0}-e_i^0+m_i^0)}-1\right)^2$$

$$+\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{m}\left(\frac{v_{ij}^{t}/X_{C_j}^{t}}{v_{ij}^{0}/X_{C_j}^{0}}-1\right)^2+\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{m}\left(\frac{v_{ij}^{t}/V_{i}^{t}}{v_{ij}^{0}/V_{i}^{0}}-1\right)^2$$

$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{l-2}\left(\frac{f_{ij}^{t}/F_{j}^{t}}{f_{ij}^{0}/F_{j}^{0}}-1\right)^2+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{l-2}\left(\frac{f_{ij}^{t}/(X_{R_i}^{t}-e_i^t+m_i^t)}{f_{ij}^{0}/(X_{R_i}^{0}-e_i^0+m_i^0)}-1\right)^2$$



s.t. :

$$\sum_{i=1}^{n}x_{ij}^{t}+\sum_{i=1}^{k}v_{ij}^{t}=X_{C_j}^{t}\quad (j=1\cdots m)$$

$$\sum_{i=1}^{n}f_{ij}^{t}=F_{j}^{t}\quad (j=1\cdots l-2)$$

$$\sum_{j=1}^{m}x_{ij}^{t}=X_{R_i}^{t}-e_i^t+m_i^t\quad (i=1\cdots n)$$

$$\sum_{j=1}^{m}v_{ij}^{t}=V_{i}^{t}\quad (i=1\cdots k)$$

* Reference

* Kuroda, M (1988), A Method of estimation for updating transaction matrix in the input-output relationships, Uno, K. and Shishido, S. eds., Statistical Data Bank Systems, Socio-Economic Database and Model Building in Japan, Amsterdam, pp.128-148.
* 宮川幸三（2001）「簡易産業連関表における未定乗数法の利用」『平成12年度産業構造の早期把握に関する調査研究（2）』機械工業経済研究報告書、pp.19-39.
* 宮川幸三（2000）「簡易産業連関表における未定乗数法の検討」『平成11年度産業構造の早期把握に関する調査研究』機械工業経済研究報告書、pp.17-34.